Brief Report

A simple scheme of teleportation of arbitrary multipartite qubit entanglement

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In this paper, we define a cross product operator and construct the cross Bell basis, by use this basis and Bell measurements we give a simple scheme of the teleportation of arbitrary multipartite qubit entanglement.

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In modern quantum mechanics and quantum information, the quantum teleportation is a quite interesting and important topic. Following the BBCJPW scheme[1], there have been very many related works (e.g. see the references in [2]). Generally, these works discussed the teleportaion of the single unknown qubit states. Recently, Rigolin[3] gives a schemes of the teleportation of arbitrary multipartite qubit states, there are yet some relational improvements (e.g., see [4]). However these schemes are more complex whatever in physics and in mathematical forms. Generally, in a perfect quantum teleportation scheme, the most basic matter are to find the quantum channels (they best form a basis of the a Hilbert space) and to give a real physical measurement way distinguishing the outcomes of the wave function collapses. In this paper, we give a simple scheme of teleportation of arbitrary multipartite qubit entanglement. Our ways are to define a convenient operator 'cross products', to construct the 'cross Bell basis' and to use two or more common Bell measurements.

In the following we write the Hilbert space of states of a spin-$\frac{1}{2}$ particle x as H_x, in which a pure-states $\mid \Psi_x\rangle = \sum\limits_{i=0,1} c_i \mid i_x\rangle$. In this paper, we mainly discuss the Hilbert space $H_1 \otimes H_2 \otimes H_3 \otimes H_4$.

Definition. Suppose that $\mid \Psi_{13}\rangle = \sum\limits_{i,j=0,1} c_{ij} \mid i_1\rangle \mid j_3\rangle \in H_1 \otimes H_3$, $\mid \Phi_{24}\rangle = \sum\limits_{r,s=0,1} d_{rs} \mid r_2\rangle \mid s_4\rangle \in H_2 \otimes H_4$ are two pure-states, then the cross product $\mid \Psi_{13}\rangle \triangledown \mid \Phi_{24}\rangle \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$ of $\mid \Psi_{13}\rangle$ and $\mid \Phi_{24}\rangle$ is defined to be the result of $\mid \Psi_{13}\rangle \otimes \mid \Phi_{24}\rangle$ returning to the natural order 1,2,3,4, i.e.

$$\mid \Psi_{13}\rangle \triangledown \mid \Phi_{24}\rangle = \sum_{i,r,j,s=0,1} c_{ij} d_{rs} \mid i_1\rangle \mid r_2\rangle \mid j_3\rangle \mid s_4\rangle \tag{1}$$

Notice that since the order of H_1, H_2, H_3, H_4 is important in our discussion, so the operations $\otimes$ and $\triangledown$ are distinct. In addition, obviously $\triangledown$ is a bi-linear and non-commutative operator.

Now, we read the ordinary Bell bases as

$$\mid \Psi^{\pm}_{\alpha\beta}\rangle = \frac{1}{\sqrt{2}} \left(\mid 0_\alpha\rangle \mid 0_\beta\rangle \pm \mid 1_\alpha\rangle \mid 1_\beta\rangle\right), \mid \Phi^{\pm}_{\alpha\beta}\rangle = \frac{1}{\sqrt{2}} \left(\mid 0_\alpha\rangle \mid 1_\beta\rangle \pm \mid 1_\alpha\rangle \mid 0_\beta\rangle\right) \tag{2}$$

then by using crossed products and according to the rule similar to matrix entries, we can write a group $\mathbb{B} = \{K \triangledown L\}$ of sixteen pure-states $K \triangledown L$ as

$$\mathbb{B} = \begin{array}{lllll} {\scriptstyle K\triangledown L} & {\scriptstyle \Psi^+_{24}} & {\scriptstyle \Psi^-_{24}} & {\scriptstyle \Phi^+_{24}} & {\scriptstyle \Phi^-_{24}} \\ {\scriptstyle \Psi^+_{13}} & \mid \Psi^+_{13}\rangle \triangledown \mid \Psi^+_{24}\rangle, & \mid \Psi^+_{13}\rangle \triangledown \mid \Psi^-_{24}\rangle, & \mid \Psi^+_{13}\rangle \triangledown \mid \Phi^+_{24}\rangle, & \mid \Psi^+_{13}\rangle \triangledown \mid \Phi^-_{24}\rangle \\ {\scriptstyle \Psi^-_{13}} & \mid \Psi^-_{13}\rangle \triangledown \mid \Psi^+_{24}\rangle, & \mid \Psi^-_{13}\rangle \triangledown \mid \Psi^-_{24}\rangle, & \mid \Psi^-_{13}\rangle \triangledown \mid \Phi^+_{24}\rangle, & \mid \Psi^-_{13}\rangle \triangledown \mid \Phi^-_{24}\rangle \\ {\scriptstyle \Phi^+_{13}} & \mid \Phi^+_{13}\rangle \triangledown \mid \Psi^+_{24}\rangle, & \mid \Phi^+_{13}\rangle \triangledown \mid \Psi^-_{24}\rangle, & \mid \Phi^+_{13}\rangle \triangledown \mid \Phi^+_{24}\rangle, & \mid \Phi^+_{13}\rangle \triangledown \mid \Phi^-_{24}\rangle \\ {\scriptstyle \Phi^-_{13}} & \mid \Phi^-_{13}\rangle \triangledown \mid \Psi^+_{24}\rangle, & \mid \Phi^-_{13}\rangle \triangledown \mid \Psi^-_{24}\rangle, & \mid \Phi^-_{13}\rangle \triangledown \mid \Phi^+_{24}\rangle, & \mid \Phi^-_{13}\rangle \triangledown \mid \Phi^-_{24}\rangle \end{array} \tag{3}$$

It is easily verified that $\mathbb{B}$ is a complete orthogonal basis of $H_1 \otimes H_2 \otimes H_3 \otimes H_4$, we call it the 'crossed Bell basis'. Here it must be stressed that these bases are really distinguishable by Bell measurements. For instance, for any state $\mid \Psi_{1234}\rangle \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$ if we make two independent Bell measurements jointed particle pair $(1,3)$ and jointed particle pair $(2,4)$ respectively, then $\mid \Psi_{1234}\rangle$ must collapse to one of the above sixteen crossed Bell bases with a probability. In the following, we notice $\overset{\vee}{x} = 1 - x$ for $x = 0$ or 1. The transformation from the natural basis to the crossed Bell basis is

$$\mid i_1\rangle \mid r_2\rangle \mid i_3\rangle \mid r_4\rangle = \frac{1}{2}\left(\mid \Psi^+_{13}\rangle + \mid \Psi^-_{13}\rangle\right) \triangledown \left(\mid \Psi^+_{24}\rangle + \mid \Psi^-_{24}\rangle\right)$$
$$\mid i_1\rangle \mid r_2\rangle \mid i_3\rangle \mid \overset{\vee}{r}_4\rangle = \frac{1}{2}(-1)^r \left(\mid \Psi^+_{13}\rangle + \mid \Psi^-_{13}\rangle\right) \triangledown \left(\mid \Phi^+_{24}\rangle + \mid \Phi^-_{24}\rangle\right)$$

$$
\begin{aligned}
\mid i_1\rangle \mid r_2\rangle \mid \overset{\vee}{i}_3\rangle \mid r_4\rangle &= \frac{1}{2}(-1)^i \left(\mid \Phi_{13}^+\rangle + \mid \Phi_{13}^-\rangle\right) \triangledown \left(\mid \Psi_{24}^+\rangle + \mid \Psi_{24}^-\rangle\right) \\
\mid i_1\rangle \mid r_2\rangle \mid \overset{\vee}{i}_3\rangle \mid \overset{\vee}{r}_4\rangle &= \frac{1}{2}(-1)^{i+r} \left(\mid \Phi_{13}^+\rangle + \mid \Phi_{13}^-\rangle\right) \triangledown \left(\mid \Phi_{24}^+\rangle + \mid \Phi_{24}^-\rangle\right)
\end{aligned} \tag{4}
$$

Now we prove that by using any one of cross Bell bases, we can realize the teleportation of a unknown bipartite qubit pure-state. For instance, we take $\mid \Phi_{13}^+\rangle \triangledown \mid \Phi_{24}^-\rangle$, as the quantum channel, particles 3, 4 are in Alice. The receiptor is Bob, she is in remote place from Alice, and she holds particles 1, 2. Suppose that $\mid \varphi_{56}\rangle = \alpha \mid 0_5 0_6\rangle + \beta \mid 0_5 1_6\rangle + \gamma \mid 1_5 0_6\rangle + \delta \mid 1_5 1_6\rangle$ is a client unknown state in Alice. It is known[5] that $\mid \psi_{56}\rangle$ is entangled if and only if $\alpha\gamma - \beta\delta \neq 0$.

In the present case the total state is

$$
\begin{aligned}
&\mid \Psi_{123456}\rangle = \left(\mid \Phi_{13}^+\rangle \triangledown \mid \Phi_{24}^-\rangle\right) \otimes \mid \varphi_{56}\rangle \\
&= \frac{1}{2}\left(\mid 0_1 0_2 1_3 1_4\rangle - \mid 0_1 1_2 1_3 0_4\rangle + \mid 1_1 0_2 0_3 1_4\rangle - \mid 1_1 1_2 0_3 0_4\rangle\right) \\
&\quad \otimes \left(\alpha \mid 0_5 0_6\rangle + \beta \mid 0_5 1_6\rangle + \gamma \mid 1_5 0_6\rangle + \delta \mid 1_5 1_6\rangle\right)
\end{aligned} \tag{5}
$$

Expanding $\mid \Psi_{123456}\rangle$ and by using the formulae in Eq.(2) to particles 3, 4, 5 and 6, the final result is

$$
\mid \Psi_{123456}\rangle = \left\{
\begin{array}{c}
\frac{1}{4}\left(\delta \mid 0_1 0_2\rangle + \gamma \mid 0_1 1_2\rangle - \beta \mid 1_1 0_2\rangle - \alpha \mid 1_1 1_2\rangle\right) \Psi_{35}^+ \triangledown \Psi_{46}^+ \\
+\frac{1}{4}\left(\delta \mid 0_1 0_2\rangle + \gamma \mid 0_1 1_2\rangle + \beta \mid 1_1 0_2\rangle - \alpha \mid 1_1 1_2\rangle\right) \Psi_{35}^+ \triangledown \Psi_{46}^- \\
+\frac{1}{4}\left(\gamma \mid 0_1 0_2\rangle + \delta \mid 0_1 1_2\rangle - \alpha \mid 1_1 0_2\rangle - \beta \mid 1_1 1_2\rangle\right) \Psi_{35}^+ \triangledown \Phi_{46}^+ \\
+\frac{1}{4}\left(-\gamma \mid 0_1 0_2\rangle + \delta \mid 0_1 1_2\rangle + \alpha \mid 1_1 0_2\rangle - \beta \mid 1_1 1_2\rangle\right) \Psi_{35}^+ \triangledown \Phi_{46}^- \\
\\
+\frac{1}{4}\left(-\delta \mid 0_1 0_2\rangle - \gamma \mid 0_1 1_2\rangle - \beta \mid 1_1 0_2\rangle - \alpha \mid 1_1 1_2\rangle\right) \Psi_{35}^- \triangledown \Psi_{46}^+ \\
+\frac{1}{4}\left(\delta \mid 0_1 0_2\rangle - \gamma \mid 0_1 1_2\rangle + \beta \mid 1_1 0_2\rangle - \alpha \mid 1_1 1_2\rangle\right) \Psi_{35}^- \triangledown \Psi_{46}^- \\
+\frac{1}{4}\left(-\gamma \mid 0_1 0_2\rangle - \delta \mid 0_1 1_2\rangle - \alpha \mid 1_1 0_2\rangle - \beta \mid 1_1 1_2\rangle\right) \Psi_{35}^- \triangledown \Phi_{46}^+ \\
+\frac{1}{4}\left(\gamma \mid 0_1 0_2\rangle - \delta \mid 0_1 1_2\rangle + \alpha \mid 1_1 0_2\rangle - \beta \mid 1_1 1_2\rangle\right) \Psi_{35}^- \triangledown \Phi_{46}^+ \\
\\
+\frac{1}{4}\left(\beta \mid 0_1 0_2\rangle + \alpha \mid 0_1 1_2\rangle - \delta \mid 1_1 0_2\rangle - \gamma \mid 1_1 1_2\rangle\right) \Phi_{35}^+ \triangledown \Psi_{46}^+ \\
+\frac{1}{4}\left(-\beta \mid 0_1 0_2\rangle + \alpha \mid 0_1 1_2\rangle + \delta \mid 1_1 0_2\rangle - \gamma \mid 1_1 1_2\rangle\right) \Phi_{35}^+ \triangledown \Psi_{46}^- \\
+\frac{1}{4}\left(\alpha \mid 0_1 0_2\rangle + \beta \mid 0_1 1_2\rangle - \gamma \mid 1_1 0_2\rangle - \delta \mid 1_1 1_2\rangle\right) \Phi_{35}^+ \triangledown \Phi_{46}^+ \\
+\frac{1}{4}\left(-\alpha \mid 0_1 0_2\rangle + \beta \mid 0_1 1_2\rangle + \gamma \mid 1_1 0_2\rangle - \delta \mid 1_1 1_2\rangle\right) \Phi_{35}^+ \triangledown \Phi_{46}^- \\
\\
+\frac{1}{4}\left(-\beta \mid 0_1 0_2\rangle - \alpha \mid 0_1 1_2\rangle - \delta \mid 1_1 0_2\rangle - \gamma \mid 1_1 1_2\rangle\right) \Phi_{35}^- \triangledown \Psi_{46}^+ \\
+\frac{1}{4}\left(\beta \mid 0_1 0_2\rangle - \alpha \mid 0_1 1_2\rangle + \delta \mid 1_1 0_2\rangle - \gamma \mid 1_1 1_2\rangle\right) \Phi_{35}^- \triangledown \Psi_{46}^- \\
+\frac{1}{4}\left(-\alpha \mid 0_1 0_2\rangle - \beta \mid 0_1 1_2\rangle - \gamma \mid 1_1 0_2\rangle - \delta \mid 1_1 1_2\rangle\right) \Phi_{35}^- \triangledown \Phi_{46}^+ \\
\left(\alpha \mid 0_1 0_2\rangle - \beta \mid 0_1 1_2\rangle + \gamma \mid 1_1 0_2\rangle - \delta \mid 1_1 1_2\rangle\right) \Phi_{35}^- \triangledown \Phi_{46}^-
\end{array}
\right\} \tag{6}
$$

We define eight 2×2 unitary matrices by

$$
\begin{aligned}
&U_{\Psi_{35}^+} = i\sigma_y, U_{\Psi_{35}^-} = -\sigma_x, U_{\Phi_{35}^+} = \sigma_z, U_{\Phi_{35}^-} = -\sigma_0 \\
&U_{\Psi_{46}^+} = \sigma_x, U_{\Psi_{46}^-} = -i\sigma_y, U_{\Phi_{46}^+} = \sigma_0, U_{\Phi_{46}^-} = -\sigma_z
\end{aligned} \tag{7}
$$

where $\sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \sigma_y = \begin{bmatrix} & -i \\ i & \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \sigma_0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ are the Pauli matrices. Now $\mid \Psi_{123456}\rangle$ can be written simply as

$$
\begin{aligned}
&\mid \Psi_{123456}\rangle = \sum\nolimits_{K=\Psi_{35}^+,\Psi_{35}^-,\Phi_{35}^+,\Phi_{35}^+} \sum\nolimits_{L=\Psi_{46}^+,\Psi_{46}^-,\Phi_{46}^+,\Phi_{46}^+} \left(\frac{1}{4} \mid \varphi_{K\triangledown L}^*\rangle \otimes (K \triangledown L)\right) \\
&\mid \varphi_{K\triangledown L}^*\rangle = U_K \otimes U_L \left(\mid \varphi_{12}\rangle\right)
\end{aligned} \tag{8}
$$

where $\mid \varphi_{12}\rangle = \mid \varphi_{56}\left(5 \longrightarrow 1, 6 \longrightarrow 2\right)\rangle$. When Alice makes a Bell measurement of particle pair $(3,5)$, and a Bell measurement of particle pair $(4,6)$ respectively, the wave function will collapse to one $\mid \varphi_{K\triangledown L}^*\rangle \otimes (K\triangledown L)$ with probability $\frac{1}{16}$ ($K\triangledown L$ is measured by Alice, simultaneously Bob obtain a corresponding state $\mid \varphi_{K\triangledown L}^*\rangle$). When Alice informs Bob of her measurement result (one $K\triangledown L$) by a classical communication, then Bob at once knows that the correct result should be

$$| \varphi_{12}\rangle = (U_K \otimes U_L)^{-1} (| \varphi^*_{K\triangledown L}\rangle) = U_L^T \otimes U_K^T (| \varphi^*_{K\triangledown L}\rangle) \tag{9}$$

where T is the transposition. So, the bipartite qubit entanglement teleportation has been completed. If we take other cross Bell basis as channel, the steps are similar.

We see that in our method the order (13 and 24) of particles in cross Bell basis is important, in fact, if we use the natural order (12 and 34), e.g. $| \Psi_{12}^{\pm}\rangle, | \Phi_{34}^{\pm}\rangle, \cdots$, etc., the product $\triangledown$ becomes common tensor product $\otimes$, and we still choose a product of them as the quantum channel, then the process in the above scheme will lead to inconveniency and difficulty.

Discussion and conclusion. If it is known that when $\alpha\gamma - \beta\delta = 0$, then $| \varphi_{56}\rangle$ must be decomposed[5] in form as $| \varphi_{56}\rangle = (a | 0_5\rangle + b | 1_5\rangle) \otimes (c | 0_6\rangle + d | 1_6\rangle)$, then obviously the above process, in fact, becomes two independent teleportation of $| \varphi_5\rangle = a | 0_5\rangle + b | 1_5\rangle$ and $| \varphi_6\rangle = c | 0_6\rangle + d | 1_6\rangle$ respectively. In addition, for three states $| \Psi_{14}\rangle = \sum_{i,j=0,1} c_{ij} | i_1\rangle | j_4\rangle \in H_1 \otimes H_4$, $| \Phi_{25}\rangle = \sum_{r,s=0,1} d_{rs} | r_2\rangle | s_5\rangle \in H_2 \otimes H_5$ and $| \Omega_{36}\rangle = \sum_{x,y=0,1} e_{xy} | x_3\rangle | y_6\rangle \in H_3 \otimes H_6$ if we define the cross product

$$| \Psi_{14}\rangle \triangledown | \Phi_{25}\rangle \triangledown | \Omega_{36}\rangle = \sum_{i,r,j,s=0,1} c_{ij} d_{rs} e_{xy} | i_1\rangle | r_2\rangle | x_3\rangle | j_4\rangle | s_5\rangle | y_6\rangle \in \bigotimes_{m=1}^{6} H_m \tag{10}$$

and construct the cross Bell basis $\{K \triangledown L \triangledown M\}$ of $\bigotimes_{m=1}^{6} H_m$, where $K, L, M =| \Psi_{14}^{\pm}\rangle, | \Phi_{14}^{\pm}\rangle$, $L =| \Psi_{25}^{\pm}\rangle, | \Phi_{25}^{\pm}\rangle$, $M =| \Psi_{36}^{\pm}\rangle, | \Phi_{36}^{\pm}\rangle$, etc., then by a similar way we can realize the teleporation of a unknown tripartite qubit state. Obviously this method can be generalized to arbitrary dimensional cases.

To sum up, by using cross Bell bases and Bell measurements we give a simple scheme of arbitrary multipartite qubit states.

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[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett., **70**(1993)1895.

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information.* Cambridge University Press (2000).

[3] G. Rigolin, Phys. Rev. A, **71**(2005)032303.

[4] Z. J. Zhang, quant-ph/0504126.

[5] C. H. Bennett, D. P. DiVincenzo, J. Smolin, and W. K. Wootters, Phys. Rev., **A54**(1996)3824.